VIBRATION OF A THIN RECTANGULAR WING OF LARGE ASPECT RATIO IN A SUPERSONIC STREAM

(VIBRATSIIA TONKOGO PRIAMOUGOL'NOGO KRYLA BOL'SHOGO UDLINENIIA V SVERKHZVUKOVOM POTOKE)

PMM Vol.22, No.6, 1958, pp.810-819

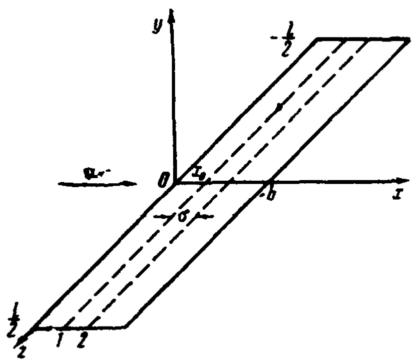
G.I. KOPZON
(Leningrad)

(Received 9 June 1955)

1. We shall consider the motion of a rectangular wing of large aspect ratio in a supersonic stream, so that the influence of the ends can, to a sufficient degree of approximation, be neglected.

Assuming that the wing is thin and that the perturbations, which clearly vary with time, are small, we can employ the expression for the velocity potential derived in [1].

Below we will solve the problem of torsional bending flutter of such a wing; in the solution we will not make the usual assumptions concerning quasi-steadiness of the stream and the exponential nature of the variation with time of the parameters under study. The figure shows the disposition of the wing in a system of coordinates fixed to it.



The equations of torsional bending deformations of the wing portrayed in the figure, when placed in a stream, can be written in the following dimensionless form:

$$\frac{\partial^4 y}{\partial z^4} - \mu_{11}\frac{\partial^2 y}{\partial \tau^2} - \mu_{12}\frac{\partial^2 \theta}{\partial \tau^2} = \varepsilon_p p(z, \tau)$$

$$-\frac{\partial^2 \theta}{\partial z^2} - \mu_{21}\frac{\partial^2 y}{\partial \tau^2} + \mu_{22}\frac{\partial^2 \theta}{\partial \tau^2} = \varepsilon_m M(z, \tau) \tag{1.1}$$

where

$$\mu_{11} = \frac{mb^4\omega^2}{EI}, \qquad \mu_{21} = \frac{m\sigma b^4\omega^2}{GI_p}, \qquad \varepsilon_p = \frac{\rho u^2}{E}\cdot\frac{b^4}{I}$$

$$\mu_{12} = \frac{m\sigma b^3\omega^2}{EI}, \qquad \mu_{22} = \frac{I_m b^2\omega^2}{GI_p}, \qquad \varepsilon_m = \frac{\rho u^2}{G}\cdot\frac{b^4}{I_p}$$

The notation is borrowed from [2]; moreover, the following dimensionless quantities are introduced:

$$y_1 = \frac{y}{b}, \qquad z_1 = \frac{z}{b}, \qquad p_1 = \frac{p}{\rho u^2 b}$$

$$M_1 = \frac{M}{\rho u^2 b^2}, \qquad \tau = \omega t$$

(the subscript 1 has been dropped in (1.1) and hereafter).

Evidently $[\omega] = \sec^{-1}$ (the quantity ω is defined below).

Expanding the deflection $y(z, t)$, the angle of torsion $\theta(z, t)$ and the normal velocity to the wing surface $v_N(x, z, t)$ in series with respect to z, we obtain

$$y(z, t) = \sum_{n=-\infty}^{\infty} a_n(t)\, e^{i\alpha_n z}$$

$$\theta(z, t) = \sum_{n=-\infty}^{\infty} b_n(t)\, e^{i\alpha_n z}, \qquad v_N(x, z, t) = \sum_{n=-\infty}^{\infty} v_n(x, t), e^{i\alpha_n z} \tag{1.2}$$

Here $\alpha_n = 2\pi n/l$, where l is the span of the wing. Evidently

$$v_n(x, t) = \dot{a}_n(t) + (x - x_0)\,\dot{b}_n(t) - ub_n(t) \tag{1.3}$$

2. In accordance with the results of [1], in the Laplace transform plane we obtain s the expressions for the dimensionless force and moment appearing in (1.1). Moreover, for the n-th harmonic of the expansion in terms of the span we will have

$$P_n(s) = [sT_0(s) + s^2T_1(s)]\,A_n(s) - [(\zeta s + 1)\,T_0(s) + \zeta s^2T_1(s) - s^2T_2(s)]\,B_n(s) -$$
$$- [sT_0(s) + s^2T_1(s)]\,a_n(0) + [\zeta sT_0(s) + s(\zeta s - 1)\,T_1(s) - s^2T_2(s)]\,b_n(0) \qquad (2.1)$$

$$M_n(s) = [\beta sT_0(s) + s(\beta s - 1)\,T_1(s) - s^2T_2(s)]\,A_n(s) - [\beta(\zeta s + 1)\,T_0(s) + \qquad (2.2)$$
$$+ (\beta\zeta s^2 - \zeta s - 1)\,T_1(s) - s^2T_2(s) + s^2T_3(s)]\,B_n(s) - [\beta sT_0(s) + s(\beta s - 1)\,T_1(s) -$$
$$- s^2T_2(s)]\,a_n(0) + [\beta\zeta sT_0(s) + s(\beta\zeta s - 1)\,T_1(s) - s(s - 1)\,T_2(s) + s^2T_3(s)]\,b_n(0)$$

(2.1) and (2.2) are obtained by means of the formulas

$$P_n(s) = \int_0^b \Delta p_n(x, s)\,dx, \qquad M_n(s) = \int_0^b (x - x_0)\,\Delta p_n(x, s)\,dx$$

$$\Delta p_n(x, s) = -\rho\left[s\Phi_n(x, 0, s) + u\frac{\partial}{\partial x}\Phi_n(x, 0, s)\right]$$

The normal velocity on the wing in the expression of the potential $\Phi_n(x, 0, s)$ is taken from formula (1.3) (see also [1]).

In what follows below we will employ the notation:

$$\zeta = \frac{x_0}{b}, \qquad \beta = 1 - \zeta, \qquad T_k(s) = \int_0^1 dx \int_0^x \cdots \int_0^x e^{\nu x} I_0(\lambda_n x)\,(dx) \qquad (2.3)$$

In addition to the dimensionless quantities already employed in the expressions (2.1), (2.2) and (2.3), we have

$$s_1 = \frac{s}{\omega}, \quad \nu_1 = \frac{M}{\sqrt{M^2 - 1}}\,\frac{bs}{a}, \quad \lambda_{n1}{}^2 = \alpha_{n1}^2 - \frac{1}{M^2 - 1}\left(\frac{bs}{a}\right)^2, \quad x_1 = \frac{x}{b}, \quad \alpha_{n1} = ba_n$$

(the subscript 1, as before, will be omitted in what follows).

Moreover, in the course of the analysis it proves to be convenient to put $\omega = u/b$; also, in formulas (2.1) and (2.2) we have $A_n(s)$, $B_n(s)$ - the transforms of $a_n(t)$, $b_n(t)$ in the Laplace plane, where $a_n(0)$, $b_n(0)$ are the initial values of $a_n(t)$, $b_n(t)$. Applying the Laplace transform to the system (1.1) and taking account of formulas (2.1), (2.2), we obtain the following system of algebraic equations for the unknown quantities $A_n(s)$, $B_n(s)$:

$$[1 + \nu_{11}s^2 - \varepsilon_p's(T_0 + sT_1)]\,A_n(s) + \{-\nu_{12}s^2 + \varepsilon_p'[(\zeta s + 1)\,T_0 + \zeta s^2T_1 - s^2T_2]\}\,B_n(s) =$$
$$= [\nu_{11}s^2 - \varepsilon_p's(T_0 + sT_1)]\,a_n(0) + \nu_{11}s\dot{a}_n(0) + \{-\nu_{12}s^2 + \varepsilon_p'[\zeta sT_0 + s(\zeta s - 1)\,T_1 -$$
$$- s^2T_2]\}\,b_n(0) - \nu_{12}s\dot{b}_n(0) \qquad (2.5)$$

$$\{\nu_{21}s^2 - \varepsilon_m'[\beta sT_0 + s(\beta s - 1)T_1 - s^2T_2]\}A_n(s) + \{1 + \nu_{22}s^2 + \varepsilon_m'[\beta(\zeta s + 1)T_0 + (\beta\zeta s^2 - \zeta s - 1)T_1 - s^2T_2 + s^2T_3]\}B_n(s) = -\{\nu_{21}s^2 + \varepsilon_m'[\beta sT_0 + s(\beta s - 1)T_1 - s^2T_2]\}a_n(0) - \nu_{21}sa_n(0) + \{\nu_{22}s^2 + \varepsilon_m'[\beta\zeta sT_0 + s(\beta\zeta s - 1)T_1 - s(s-1)T_2 + s^2T_3]\}b_n(0) + \nu_{22}s^2\dot{b}_n(0) \tag{2.6}$$

where $T_k = T_k(s)$ $(k = 0, 1, 2, 3)$; from (2.3) and (2.4) it follows that these are holomorphic functions in the s-plane.

In equations (2.5) and (2.6) the following notation is used:

$$\nu_{11} = \frac{\mu_{11}}{\alpha_n^4}, \qquad \nu_{12} = \frac{\mu_{12}}{\alpha_n^4}, \qquad \nu_{21} = \frac{\mu_{21}}{\alpha_n^2}, \qquad \nu_{22} = \frac{\mu_{22}}{\alpha_n^2}$$
$$\varepsilon_p' = \frac{\varepsilon_p}{\alpha_n^4} = \frac{\rho u^2}{E}\frac{l^4}{(2\pi n)^4 I}, \qquad \varepsilon_m' = \frac{\varepsilon_m}{\alpha_n^2} = \frac{\rho u^2}{G}\frac{b^2l^2}{(2\pi n)^2 I_p} \tag{2.7}$$

Later we will make use of the relation $\epsilon_p' = g\epsilon_m'$, where

$$g = \frac{1}{(2\pi n)^2}\left(\frac{l}{b}\right)^2\frac{I_p}{I}\frac{G}{E} \tag{2.8}$$

In order to obtain the required functions $a_n(t)$, $b_n(t)$ it is necessary to find from (2.5) and (2.6) the transforms $A_n(s)$ and $B_n(s)$ and then transform back to the original functions using the Riemann-Mellin transformation formula.

This operation is rather cumbersome; however, certain qualitative conclusions can be drawn without directly calculating $a_n(t)$ and $b_n(t)$. Thus, from consideration of (2.5) and (2.6), it can be concluded that:

(a) judging by the nature of the dependence of T_k upon s, the Riemann-Mellin integrals for a_n and b_n are determined from a single calculation (see the characteristic frequency equation below). Consequently, in the exact solution the dependence upon time under our restrictions must be of an exponential nature;

(b) ϵ_p', ϵ_m' [see (2.7)] show that the influence of the stream is important only for the lower modes. For the higher modes, by virtue of the relations $\epsilon_p' \quad 1/n^4$, $\epsilon_m' \quad 1/n^2$, the influence of the stream becomes negligibly small. It is obviously sufficient to carry out an investigation of stability in a stream for the lowest modes of the expansions (1.2) only.

It is not difficult to write down the equation for the determinant of the systems (2.5) and (2.6):

$$\Delta(s) = [1 + \nu_{11}s^2 - g\varepsilon s(T_0 + sT_1)]\{1 + \nu_{22}s^2 + \varepsilon[\beta(\zeta s + 1)T_0 + (\beta\zeta s^2 - \zeta s - 1)T_1 - s^2T_2 + s^2T_3]\} + \{\nu_{21}s^2\varepsilon[\beta sT_0 + s(\beta s - 1)T_1 - s^2T_2]\}\{-\nu_{12}s^2 + g\varepsilon[(\zeta s + 1)T_0 + \zeta s^2T_1 - s^2T_2]\} = 0 \qquad (2.9)$$

where $\epsilon = \epsilon_m'$. In the limiting case $\epsilon = 0$, we obtain

$$\Delta(s) = \Delta_0(s) = (1 + \nu_{11}s^2)(1 + \nu_{22}s^2) - \nu_{21}\nu_{12}s^4 = 0 \qquad (2.10)$$

This last equation is obviously the frequency equation in the case of vibration of the wing *in vacuo*. By analogy with it, we shall refer to (2.9) as the characteristic frequency equation for vibrations of the wing in a supersonic stream.

3. Equation (2.9) can be re-written in the following form:

$$\Delta(s) = \Delta_0(s) + \varepsilon K_1(s) + \varepsilon^2 K_2(s) = 0 \qquad (3.1)$$

where

$$\begin{aligned}
K_1(s) &= (1 + \nu_{11}s^2)D_2(s) - (1 + \nu_{22}s^2)D_1(s) + s^2[\nu_{21}D_4(s) - \nu_{12}D_3(s)] \\
K_2(s) &= D_3(s)D_4(s) - D_1(s)D_2(s) \\
D_1(s) &= gs(T_0 + sT_1) \\
D_2(s) &= \beta(\zeta s + 1)T_0 + (\beta\zeta s^2 - \zeta s - 1)T_1 - s^2T_2 + s^2T_3 \\
D_3(s) &= \beta sT_0 + s(\beta s - 1)T_1 - s^2T_2 \\
D_4(s) &= g[(\zeta s + 1)T_0 + \zeta s^2T_1 - s^2T_2]
\end{aligned}$$

Restricting consideration to the case of practical interest of a very rigid wing, i.e. neglecting ϵ^2 in comparison with ϵ, we obtain the approximate frequency equation in the form*

$$\Delta_0(s) + \varepsilon K_1(s) = 0 \qquad (3.2)$$

If a solution is now sought in the form

$$s_j = s_j^\circ + \varepsilon s_j' \qquad (j = 1, 2, 3, 4) \qquad (3.3)$$

then, neglecting terms proportional to ϵ^2, for the coefficients s_j we obtain the formula

$$s_j' = -\frac{K_1(s_j^\circ)}{[\partial\Delta_0(s)/\partial s]\, s = s_j^\circ} \qquad (3.4)$$

Here s_j° are the frequencies corresponding to the case of vibration of the wing *in vacuo*: $\Delta_0(s_j^\circ) = 0$.

* In this paper we do not make an estimate of the values of ϵ for which the described method of approximate solution of the problem remains valid.

Carrying out the algebra, we eventually obtain the following expressions:

$$s_{1,2}^{\,0} = \pm i\omega_1, \qquad s_{3,4}^{\,0} = \pm i\omega_2 \tag{3.5}$$

$$s_{1,2}' = \pm \frac{i}{2\omega_1}\frac{\Omega(\omega_1)}{D}, \qquad s_{3,4}' = \mp \frac{i}{2\omega_2}\frac{\Omega(\omega_2)}{D} \tag{3.6}$$

$$\Omega(\omega) = (1 - \nu_{11}\omega^2) D_2(\pm i\omega) - (1 - \nu_{22}\omega^2) D_1(\pm i\omega) - \omega^2 [\nu_{21} D_4(\pm i\omega) - \nu_{12} D_3(\pm i\omega)]$$

$$\omega_{1,2} = \left(\frac{1}{2\delta}[\nu_{11} + \nu_{22} \pm D]\right)^{1/2}, \qquad D = \sqrt{(\nu_{11} - \nu_{22})^2 + 4\nu_{21}\nu_{12}}$$

$$\delta = \nu_{11}\nu_{22} - \nu_{21}\nu_{12} > 0 \qquad \text{(condition for stability)}$$

In the majority of cases we can restrict consideration merely to the study of the solutions so obtained from the point of view of their stability with respect to time. In so far as the quantities $s_j^{\,0}$ are purely imaginary, the conditions of stability in the first approximation ($\epsilon^2 \approx 0$) take the form:

$$\operatorname{Re} s_j' \leqslant 0 \qquad (j = 1, 2, 3, 4) \tag{3.7}$$

By means of elementary transformations it is not difficult to deduce that (3.7) can in the final analysis be reduced to the following two conditions:

$$\operatorname{Im}\{[A(\nu_{ik}, \omega_1) + iB(\nu_{ik}, \omega_1)] T_0(i\omega_1) + [C(\nu_{ik}, \omega_1) + iD(\nu_{ik}, \omega_1)] T_1(i\omega_1) + $$
$$+ E(\nu_{ik}, \omega_1) T_2(i\omega_1) - G(\nu_{ik}, \omega_1) T_3(i\omega_1)\} \geqslant 0 \tag{3.8}$$

$$\operatorname{Im}\{[A(\nu_{ik}, \omega_2) - iB(\nu_{ik}, \omega_2)] T_0(-i\omega_2) + [C(\nu_{ik}, \omega_2) - iD(\nu_{ik}, \omega_2)] T_1(-i\omega_2) + $$
$$+ E(\nu_{ik}, \omega_2) T_2(-i\omega_2) - G(\nu_{ik}, \omega_2) T_3(-i\omega_2)\} \geqslant 0 \tag{3.9}$$

In (3.8) and (3.9) the following notation is used:

$$A(\nu_{ik}, \omega) = \beta(1 - \nu_{11}\omega^2) - g\nu_{21}\omega^2$$

$$B(\nu_{ik}, \omega) = -\omega g(1 - \nu_{22}\omega^2) + \beta\omega\zeta(1 - \nu_{11}\omega^2) + \omega^3\beta\nu_{12} - g\nu_{21}\omega^2$$

$$C(\nu_{ik}, \omega) = \omega^2 g(1 - \nu_{22}\omega^2) - (\beta\zeta\omega^2 + 1)(1 - \nu_{11}\omega^2) - \omega^4\beta\nu_{12}$$

$$D(\nu_{ik}, \omega) = -\omega\zeta(1 - \nu_{11}\omega^2) - \omega^3\nu_{12}$$

$$E(\nu_{ik}, \omega) = \omega^2(1 - \nu_{11}\omega^2) + \omega^4(\nu_{12} - \nu_{21})$$

$$G(\nu_{ik}, \omega) = \omega^2(1 - \nu_{11}\omega^2)$$

It is interesting to observe that the expressions appearing on the left of (3.8) and (3.9) contain the mechanical characteristics of the structure of the wing (the quantities ν_{ik}, $\omega_{1,2}$) and the parameters of the unperturbed stream (in terms of T_k). Accordingly, the conditions (3.8) and (3.9) can serve as criteria of the stability of a given wing structure in a supersonic stream. In the computations it is convenient to make use of the tables of the functions

$$T(a,\, b) = \int_0^a \exp i\sigma I_0 (b\sigma)\, d\sigma$$

appearing, for example, in [3]. From these it is easy to calculate the functions $T_k(\pm\, i\omega_{1,\,2})$ which are of interest here.

4. Conditions (3.8) and (3.9) are useful as criteria for the stability in the stream of a given wing structure. For variational analysis of the structural parameters they appear to be unsuitable in so far as they contain quantities which are defined in tables. Let us write down approximate expressions for the functions $T_k(i\omega)$ in explicit analytical form. For this purpose, we will start from the last of the formulas (2.3) and the integral representation of the zero-order Bessel function in the form

$$I_0 (\lambda x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda x \sin\theta}\, d\theta \qquad (4.1)$$

Substituting (4.1) in (2.3), we eventually obtain

$$T_k (s) = \sum_{n=0}^{\infty} \frac{1}{(n+k+1)!} \int_{-\pi}^{\pi} (\nu + i\lambda \sin\theta)^n\, d\theta \qquad (4.2)$$

Using the fact that

$$\int_{-\pi}^{\pi} \sin^{2r}\theta\, d\theta = \frac{2\pi\ (2r-1)!!}{2^r r!}$$

we obtain the following expressions:

$$T_k (s) = \sum_{n=0}^{\infty} \frac{C_n(s)}{(n+k+1)!}\, \nu^n, \qquad C_n(s) = 2\pi \sum_{r=0}^{r\leqslant {}^1\!/_2 n} (-1)^r \frac{(2r-1)!!}{r!} \binom{n}{2r} \left(\frac{\lambda^2}{2\nu^2}\right)^r \qquad (4.3)$$

The expressions $\lambda = \lambda_n$ and ν are given by the formulas (2.4). Substituting in (4.3) $s = i\omega$, we obtain

$$C_n (i\omega) = 2\pi \sum_{r=0}^{r\leqslant {}^1\!/_2 n} \binom{n}{2r} \frac{(2r-1)!!}{r!} \left(\frac{\xi}{\sqrt{2}}\right)^{2r}, \qquad \xi = \sqrt{\frac{(M^2-1)\, a^2\alpha^2 + \omega^2}{M^2\omega^2}} > 0 \qquad (4.4)$$

Obviously, we can write

$$\frac{(2r-1)!!}{r!} = \prod_{k=1}^{r} \frac{2(r-k)+1}{r-k+1}, \qquad 1 \leqslant \frac{2(r-k)+1}{(r-k)+1} < 2$$

so that

$$1 \leqslant \frac{(2r-1)!!}{r!} < 2^r$$

Substituting the last inequality in expression (4.4) and carrying out the summation with respect to r, and remembering that all terms of (4.4) are positive, we obtain the following estimates:

$$\min[C_n(i\omega)] = \pi\left[\left(1+\frac{\xi}{\sqrt{2}}\right)^n - \left(1-\frac{\xi}{\sqrt{2}}\right)^n\right]$$

$$\max[C_n(i\omega)] = \pi[(1+\xi)^n - (1-\xi)^n] \qquad (4.5)$$

Putting

$$C_n(i\omega) = \pi[(1+\zeta)^n - (1-\zeta)^n] \qquad \left(\frac{\xi}{\sqrt{2}} \leqslant \zeta < \xi\right)$$

we will have

$$T_k(i\omega) = \pi \sum_{n=0}^{\infty} \frac{1}{(n+k+1)!}\{[i\omega(1+\zeta)]^n - [i\omega(1-\zeta)]^n\}$$

In the last formula let us carry out the infinite summation to obtain the final result:

$$T_k(i\omega) = \frac{\pi}{[i\omega(1+\zeta)]^{k+1}}\left\{e^{i\omega(1+\zeta)} - \sum_{q=0}^{k}\frac{1}{q!}[i\omega(1+\zeta)]^q\right\} - \\ - \frac{\pi}{[i\omega(1-\zeta)]^{k+1}}\left\{e^{i\omega(1-\zeta)} - \sum_{q=0}^{k}\frac{1}{q!}[i\omega(1-\zeta)]^q\right\} \qquad (4.6)$$

In view of the fact that $\zeta_{max} = \sqrt{2}\,\zeta_{min} \approx 1.41\,\zeta_{min}$, we can make use of expression (4.6) by putting $\zeta \approx \zeta_{av} \approx 1.2\,\zeta_{min}$.

In view of the fact that the values of k which are of interest to us are not large (k = 0, 1, 2, 3), the approximate expressions (4.6) are not very cumbersome when used in conditions (3.8) and (3.9).

After the final choice of all the parameters of the wing it is advisable to evaluate the criterion of stability by means of the tabulated values of the functions $T_k(i\omega)$.

BIBLIOGRAPHY

1. Kopzon, G.I., Nestatsionarnye dvizheniia kryla v sverkhzvukovom potoke (Unsteady motion of a wing in a supersonic stream). *PMM* Vol. 21, No. 1, 1957.

2. Grossman, E.P., Flatter (Flutter). *Trudy TsAGI* No. 284, 1937.

3. Schwarz, L., Untersuchung einiger mit den Zylinderfunktionen nullter Ordnung verwandter Funktionen. *Luftfahrtforsch.* Vol. 20, pp. 340-372, 1944.